University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

SOLUTION TO EXAMINATION 2

Directions. Do all four problems (weights are indicated). This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Calculators are allowed but not essential – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (25 points)
The basis of scalar diffraction theory is the *Fresnel-Kirchoff integral formula*. In Fowles' notation (Eq. 5.11), this formula states

$$U_{\mathrm{p}} = -\frac{ikU_0 \exp(-i\omega t)}{4\pi} \times \\ \times \int\int \frac{\exp\big(ik(r+r')\big)}{rr'} \left[\hat{\mathbf{n}}\cdot\hat{\mathbf{r}} - \hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'\right] d\mathcal{A}$$

where $\mathbf{r}'$ is a vector from the (point) source to a point on the aperture, $\mathbf{r}$ is a vector from the observer to the same point on the aperture,

$$U_0 \frac{\exp\big(i(kr' - \omega t)\big)}{r'}$$

is the optical disturbance at a point on the aperture, U_{p} is the optical disturbance at the observer, $\omega = ck = 2\pi c/\lambda$ is the angular frequency of the light, $d\mathcal{A}$ is an element of aperture area, and $\hat{\mathbf{n}}$ is the normal to $d\mathcal{A}$. [Note that, in a typical geometry (source on the left, aperture in the middle, observer on the right, and $\hat{\mathbf{n}}$ pointing to the left), $\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}$ is positive while $\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'$ is negative, so that both terms in the square bracket are positive.]

Consider this simple geometry: Let z be the axis pointing from left to right. Place the source at $(x, y, z) = (0, 0, -D)$, the observer at (X, Y, D), and the aperture in the plane $z = 0$. The aperture is characterized by an *aperture function* $g(x, y)$ such that $g = 1$ where the aperture is open, and $g = 0$ where the aperture is opaque.

a. (10 points) Let δ be the maximum value of $\sqrt{x^2+y^2}$ on the aperture plane for which the aperture is *not* opaque. Thus, for this part of the problem, there are three characteristic lengths: λ, δ, and D. By moving around in the plane $z = D$, restricting her own coordinates X, Y such that $\sqrt{X^2+Y^2} \ll D$, the observer finds that the optical disturbance there is proportional to the *Fourier transform* of $g(x, y)$. As someone who understands the physics of diffraction, you realize that this information implies that a single strong condition must be satisfied which relates λ, δ, and D. *Write down this condition.* (You needn't prove it, and you may omit factors of order unity.)

Solution. In order for the optical disturbance $U_p(X, Y)$ to be the Fourier transform of $g(x, y)$, our system must satisfy the Fraunhofer condition (see discussion in Fowles Section 5.6). The basic idea of this condition is that the spherical curvature of the wavefront at the aperture must be small compared to the wavelength of the light, allowing us to treat the light at the aperture as a plane wave. Omitting factors of order unity, the Fraunhofer condition is

$$\delta^2 \ll D\lambda\ .$$

This condition implies that the obliquity factor $\left[\hat{\mathbf{n}}\cdot\hat{\mathbf{r}} - \hat{\mathbf{n}}\cdot\hat{\mathbf{r}}'\right]$ is constant over the aperture, the quantity $e^{ikr'}/r'$ is nearly constant, and the quantity $e^{ikr}/r \approx e^{ikr}$. With these approximations, the optical disturbance

$$U_p(X, Y) \propto \int\int e^{ikr} d\mathcal{A}.$$

b. (15 points) For this part of the problem, take the aperture function to be

$$g(x, y) = 0, \quad x < 0$$
$$g(x, y) = 1, \quad x > 0 \; .$$

This describes a "knife edge" at $x = 0$ extending from $y = -\infty$ to $y = \infty$. Therefore, in this part of the problem, $\delta = \infty$: the strong condition of part **a.** cannot be satisfied. In this part of the problem, the observer is fixed at $(0, 0, D)$, *i.e.* at $X = Y = 0$. With this aperture in place, the observer records an *irradiance* I_a. With the aperture completely removed ($g \equiv 1$), the observer records an irradiance I_0. Give the ratio I_a/I_0. To receive credit you must *explain why this ratio is correct.*

Solution. According to the Fresnel-Kirchoff integral, the optical disturbance U_p arises from a superposition of secondary waves which originate at the $z = 0$ plane. When the semi-infinite screen is in place, due to the symmetry of the system exactly half the secondary waves are blocked. Thus

$$U_p = \frac{1}{2} U_0 \; ,$$

or, in terms of intensity $I \propto |U|^2$,

$$\frac{I}{I_0} = \frac{1}{4} \; .$$

2. (25 points)
James Rainwater was awarded the Nobel Prize in the 1980's for experiments done at the Nevis (Columbia) cyclotron in the 1950's. He measured the sizes of nuclei using their interactions with muons (heavy electrons) which were in orbit about them.

In the following, use the Bohr picture to describe the muon orbit. For ease of numerical computation, you may take the natural length unit $\hbar/m_e c$ to be 400 fm; the ratio m_μ/m_e of muon to electron masses to be 200; and the fine structure constant α to be 1/150. You may neglect the difference between the muon's actual and reduced mass.

A muon in $n = 1$ Bohr orbit reacts with (is "captured" by) a $Z = 50$ nucleus before it decays:

$$\mu^- + (A, Z) \rightarrow (A, Z - 1) + \nu_\mu ,$$

where the neutrino ν_μ has negligible rest mass. Assuming that the initial and final nuclei have the same infinitely large rest mass and therefore a negligible kinetic energy, what is the neutrino energy expressed in units of $m_e c^2$? (1% accuracy is sufficient.)

Solution. The binding energy of the muon in the Bohr model is given by:

$$BE = \frac{1}{2} m_\mu c^2 (Z\alpha)^2$$

and in our case $Z\alpha \approx 1/3$ and $m_\mu c^2 \approx 200 m_e c^2$. So

$$BE \approx 11 m_e c^2 \; .$$

Since the rest energies of the initial and final nuclei are taken to be the same, the kinetic energy of the neutrino must be equal to the rest mass of the muon minus the binding energy, or

$$KE(\nu_\mu) \approx 200 m_e c^2 - 11 m_e c^2 \approx 189 m_e c^2 \; .$$

3. (25 points)
Consider the elastic scattering of a photon from an infinitely massive, perfectly reflective, spherical target of finite radius R (like a bowling ball polished to a mirror finish). The bowling ball is centered on the origin. The photon is incident along the $\hat{\mathbf{z}}$ direction and scatters (reflects) into the direction (θ, ϕ), where θ and ϕ are the usual spherical polar angles. Note that $\theta = 0$ means that the photon remains undeflected. For this problem, ignore diffraction and any other effects which arise from the wavelike properties of the photon.

a. (10 points) What is the total scattering cross section σ_T, corresponding to any deflection of the photon? (You don't need a calculation here, just a correct answer and a convincing explanation for it.)

Solution. The total cross section σ_T is the cross sectional area of the photon beam that suffers *any* deflection as a result of interaction with the target. Neglecting diffractive effects, the only photons scattered are those which intercept the area of a hemisphere of radius R, projected into the $z = 0$ plane. This is a circle of area πR^2. Thus

$$\sigma_T = \pi R^2 \ .$$

b. (15 points) Calculate the differential cross section

$$\frac{d\sigma}{d\Omega} \ ,$$

where $d\Omega = \sin\theta \, d\theta \, d\phi$ is an element of solid angle. (When you integrate your result over the full solid angle, do you confirm your answer to **a.**?)

Solution. A photon with impact parameter $b = \sqrt{x^2 + y^2}$ intercepts the sphere at a point on the sphere described by

$$\theta_s = \pi - \arcsin\frac{b}{R} \equiv \pi - \psi \ .$$

Just before it hits the sphere, it is travelling in the direction

$$\theta_0 = 0 \ .$$

Before impact, the angle that the photon makes with the normal to the sphere is ψ. Since the angle of incidence is equal to the angle of reflection, its direction changes by

$$\Delta\theta = \pi - 2\psi \ .$$

Therefore the final angle θ of the photon is

$$\begin{aligned} \theta &= \theta_0 + \Delta\theta \\ &= 0 + \pi - 2\psi \\ &= \pi - 2\arcsin\frac{b}{R} \ . \end{aligned}$$

Rearranging and differentiating,

$$\begin{aligned} \arcsin\frac{b}{R} &= \frac{\pi}{2} - \frac{\theta}{2} \\ b &= R\,\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\ &= R\,\cos\frac{\theta}{2} \\ db &= -\frac{R}{2}\,\sin\frac{\theta}{2}\,d\theta \ . \end{aligned}$$

An element $d\sigma$ of beam cross section is equal to $|b\,db\,d\phi|$. Substituting from above,

$$\begin{aligned} d\sigma &= |b\,db\,d\phi| \\ &= \frac{R^2}{2}\,\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}\,d\theta\,d\phi \\ &= \frac{R^2}{4}\,\sin\theta\,d\theta\,d\phi \\ &= \frac{R^2}{4}\,d\Omega \\ \frac{d\sigma}{d\Omega} &= \frac{R^2}{4} \ . \end{aligned}$$

This is an isotropic (constant) differential cross section. Integrated over $\Delta\Omega = 4\pi$, it yields $\sigma_T = \pi R^2$ as in (**a.**). Note that the isotropy of the differential cross section doesn't follow automatically from the spherical symmetry of the potential (an infinite wall at $r = R$). A different spherically symmetric potential, for example the Coulomb potential, yields the dramatically different Rutherford result

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\frac{\theta}{2}} \ .$$

4. (25 points)
A nonrelativistic particle of mass m is confined to a one-dimensional box extending from $x = 0$ to $x = L$. Here a "box" is a square potential well with infinite sides.

a. (10 points) In terms of n and other constants, write down the energies E_n, $1 \le n < \infty$, measured with respect to the bottom of the potential well, that the particle is allowed by Schrödinger's equation to have.

Solution. We measure the energy E of the particle with respect to the bottom of the well, where $V \equiv 0$. We seek solutions of the time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)u_E(x) = E u_E(x) \ ,$$

with the boundary condition (because of the infinite potential wall)

$$u_E(0) = u_E(L) = 0 \ .$$

The solutions are of the form

$$u_E(x) \propto \sin k_n x$$

with $k_n = n\pi/L$. Therefore, from the time-independent Schrödinger equation,

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2} \ ,$$

with $1 \le n \le \infty$. [As posed, the problem doesn't require a proof like the above; you just need to write down the correct values of E_n.]

b. (15 points) Define $N(E)$ to be the total number of allowed states with energy $\le E$. Taking $n \gg 1$, so that the distribution of E is approximately continuous, calculate the density of states

$$\rho(E) \equiv \frac{dN}{dE} \ .$$

Solution. The difference in energy between two adjacent states is

$$\begin{aligned}
\Delta E &\equiv E_n - E_{n-1} \\
&= \left(n^2 - (n-1)^2\right)\frac{\pi^2\hbar^2}{2mL^2} \\
&= (2n-1)\frac{\pi^2\hbar^2}{2mL^2} \ .
\end{aligned}$$

So when we increase the number of states by $\Delta N = 1$ we increase the maximum energy by ΔE. The density of states is just the ratio:

$$\begin{aligned}
\rho(E) &\equiv \frac{dN}{dE} \\
&\approx \frac{\Delta N}{\Delta E} \text{ as } n \to \infty \\
&= \frac{2mL^2}{(2n-1)\pi^2\hbar^2} \\
&\approx \frac{mL^2}{n\pi^2\hbar^2} \text{ as } n \to \infty \ .
\end{aligned}$$

Expressing $\rho(E)$ in terms of E and other constants, we substitute

$$\begin{aligned}
n^2 &= \frac{2mL^2E}{\pi^2\hbar^2} \\
\rho(E) &= \frac{mL^2}{\pi^2\hbar^2}\sqrt{\frac{\pi^2\hbar^2}{2mL^2E}} \\
&= \frac{L}{\pi\hbar}\sqrt{\frac{m}{2E}} \ .
\end{aligned}$$